

# A velocity-dependent potential of a rigid body in a rotating frame

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We derive a velocity-dependent potential for describing the dynamics of a rigid body in a rotating frame. We show that, as for one-particle systems, the different components of this potential can be associated with electromagnetic analogs. We provide some examples to demonstrate the feasibility of using the potential as an alternative description of rigid body problems. © 2008 American Association of Physics Teachers.

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## I. INTRODUCTION

Students in classical dynamics courses are taught how to treat a reference frame rotating with angular velocity  $\boldsymbol{\omega}$  by including three inertial forces<sup>1</sup> in addition to real physical forces in Newton's equation,  $m\ddot{\mathbf{r}}=\mathbf{F}$ . For a particle of mass  $m$  and position  $\mathbf{r}(t)$  these inertial forces are the Coriolis force,<sup>2</sup>

$$\mathbf{F}_{\text{Coriolis}} = -2m\boldsymbol{\omega} \times \dot{\mathbf{r}}, \quad (1)$$

the centrifugal force

$$\mathbf{F}_{\text{centrifugal}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \quad (2)$$

and the Euler force

$$\mathbf{F}_{\text{Euler}} = -m\dot{\boldsymbol{\omega}} \times \mathbf{r}. \quad (3)$$

These inertial forces are not essential and it has been suggested that they should not be discussed in introductory courses (see, for instance, Ref. 6). However, after considering a few examples and problems including the Foucault pendulum, projectile deflection, and hurricanes, students come to appreciate that the use of inertial forces can lead to a much simpler and more intuitive description of rotating systems.<sup>7</sup>

In spite of its advantages most textbook authors, with a few exceptions,<sup>8</sup> have avoided using inertial forces within Lagrangian mechanics by introducing velocity-dependent potentials. Such a generalization is simple enough<sup>9</sup> and can be used to ease the description of rotating systems that are difficult to analyze when studied from an inertial frame of reference. Furthermore, its understanding can be eased by a one-to-one analogy with a particular electromagnetic vector potential in a symmetric gauge.<sup>9</sup>

## II. VELOCITY-DEPENDENT POTENTIAL

We consider the Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad (j = 1, 3N - k), \quad (4)$$

for a system of  $N$  particles of mass  $m_i$  and  $k$  holonomic constraints. The position  $\mathbf{r}_i(t)$  of each particle can be expressed in terms of the  $3N - k$  generalized coordinates  $q_j$  as  $\mathbf{r}_i(q_1, \dots, q_{3N-k}, t)$ . The kinetic energy  $T$  in Eq. (4) is defined as

$$T = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2. \quad (5)$$

The generalized forces are given by

$$Q_j = \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \quad (6)$$

If part of these forces can be derived from a potential energy function  $V$  that depends only on the coordinates  $q_j$ , the velocities  $\dot{q}_j$ , and the time  $t$  as

$$Q_j = \tilde{Q}_j + \frac{d}{dt} \left( \frac{\partial V}{\partial \dot{q}_j} \right) - \frac{\partial V}{\partial q_j}, \quad (7)$$

the Lagrange equations can be rewritten as

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = \tilde{Q}_j, \quad (8)$$

where  $\mathcal{L} = T - V$  is the Lagrangian function and  $\tilde{Q}_j$  is the part of the generalized force which cannot be written in terms of a potential function. In what follows, we shall assume  $\tilde{Q}_j = 0$ .

Velocity-dependent potentials were introduced into Lagrangian mechanics in 1873 by the mathematician Schering (1833–1897), Ref. 10 as a way of dealing with the pre-Maxwellian electrodynamic theory of Weber (1804–1891).<sup>11</sup> It was called the Schering potential by Whittaker (1873–1956) in the first edition of his analytical dynamics text,<sup>12</sup> but he dropped this attribution in later editions (see Ref. 13).

## III. VELOCITY-DEPENDENT POTENTIAL OF AN ELECTROMAGNETIC FIELD

A velocity-dependent potential can help solve inverse problems<sup>14</sup> and is essential for treating a particle of charge  $q$  in the presence of an electromagnetic field  $\mathbf{E}(\mathbf{r}, t)$ ,  $\mathbf{B}(\mathbf{r}, t)$ . The Lorentz force  $\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$  can be written in terms of the scalar  $\phi$  and vector  $\mathbf{A}$  potentials (such that  $\mathbf{B} = \nabla \times \mathbf{A}$  and  $\mathbf{E} = -\nabla \phi - \partial \mathbf{A} / \partial t$ ) as

$$\mathbf{F} = q \left( -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \dot{\mathbf{r}} \times \nabla \times \mathbf{A} \right). \quad (9)$$

From Eq. (7) we see that the following velocity-dependent potential will generate Eq. (9):

$$V = q\phi - q\dot{\mathbf{r}} \cdot \mathbf{A}. \quad (10)$$

#### IV. INERTIAL POTENTIAL FOR POINTLIKE PARTICLES

We consider a particle of mass  $m$  at a position  $\mathbf{r}$  in a reference frame rotating with velocity  $\boldsymbol{\omega}(t)$ . Some simple algebra shows that the three fictitious forces Eqs. (1)–(3) can be rewritten as

$$\mathbf{F}_I = \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Euler}} + \mathbf{F}_{\text{Coriolis}} \quad (11a)$$

$$\begin{aligned} &= \frac{m}{2} \nabla (\boldsymbol{\omega} \times \mathbf{r})^2 - m \frac{\partial}{\partial t} (\boldsymbol{\omega} \times \mathbf{r}) \\ &+ m\dot{\mathbf{r}} \times (\nabla \times (\boldsymbol{\omega} \times \mathbf{r})). \end{aligned} \quad (11b)$$

This expression is identical to that of the Lorentz force Eq. (9) if the mass is identified with the charge,  $q \leftrightarrow m$ , and the following scalar and vector potentials are defined.<sup>1,9,15</sup>

$$\phi = -\frac{1}{2} (\boldsymbol{\omega} \times \mathbf{r})^2 \quad (12a)$$

$$\mathbf{A} = \boldsymbol{\omega} \times \mathbf{r}. \quad (12b)$$

For example, if  $\mathbf{B}$  is uniform and the symmetric gauge is used so that  $\mathbf{A} = -\mathbf{r} \times \mathbf{B}/2$ , we have the analogy  $\mathbf{B} = 2\boldsymbol{\omega}$ .<sup>9,16</sup> The corresponding electric field has a particular linear dependence on the position  $\mathbf{r}$ ,

$$\mathbf{E} = -4\mathbf{B} \times (\mathbf{B} \times \mathbf{r}) - 2\dot{\mathbf{B}} \times \mathbf{r} \quad (13a)$$

$$= 4B^2\mathbf{r} - 4(\mathbf{B} \cdot \mathbf{r})\mathbf{B} - 2\dot{\mathbf{B}} \times \mathbf{r}. \quad (13b)$$

We substitute Eq. (12) into Eq. (10) and obtain the following inertial potential:

$$V_I = -\frac{m}{2} (\boldsymbol{\omega} \times \mathbf{r})^2 - m\dot{\mathbf{r}} \cdot (\boldsymbol{\omega} \times \mathbf{r}). \quad (14)$$

Thus, the Lagrangian for a system of  $N$  particles of masses  $m_i$  and positions  $\mathbf{r}_i(t)$  reads<sup>8</sup>

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2 + \frac{1}{2} \sum_{i=1}^N m_i (\boldsymbol{\omega} \times \mathbf{r}_i)^2 + \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &- V_{\text{int}} \end{aligned} \quad (15a)$$

$$= \frac{1}{2} \sum_{i=1}^N m_i (\dot{\mathbf{r}}_i + \boldsymbol{\omega} \times \mathbf{r}_i)^2 - V_{\text{int}}, \quad (15b)$$

where  $\boldsymbol{\omega} \times \mathbf{r}_i$  is the velocity of particle  $i$  in the rotating reference frame,<sup>9</sup> and  $V_{\text{int}}$  represents any other possible interaction. These equations can be written in terms of general coordinates  $\{q_j\}_{j=1}^{3N-k}$  once the relations  $\mathbf{r}_i(q_1, \dots, q_{3N-k}, t)$  are established.

#### V. INERTIAL POTENTIAL FOR A RIGID BODY

Let us now apply these ideas to a rigid body consisting of  $N$  point masses  $m_i$  located at  $\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R}$  in a reference frame rotating with velocity  $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ ;  $\mathbf{R}$  is the position of the center of mass. The corresponding inertial potential is

$$V_I = -\frac{1}{2} \sum_{i=1}^N m_i (\boldsymbol{\omega} \times \mathbf{r}_i)^2 - \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot (\boldsymbol{\omega} \times \mathbf{r}_i). \quad (16)$$

We replace  $\mathbf{r}_i = \mathbf{r}'_i + \mathbf{R}$  and  $\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\Omega} \times \mathbf{r}'_i$ , where  $\boldsymbol{\Omega}$  is the angular velocity of the rigid body in the rotating reference frame, and apply the relation

$$\sum_i m_i (\mathbf{A} \times \mathbf{r}'_i) \cdot (\mathbf{B} \times \mathbf{r}'_i) = \mathbf{A} \cdot \mathcal{I} \cdot \mathbf{B}, \quad (17)$$

where  $\mathcal{I}$  is the inertial tensor with respect to the center of mass with elements  $\mathcal{I}_{jk} = \sum_{\alpha=1}^N m_{\alpha} (r'_{\alpha}{}^2 \delta_{jk} - r'_{\alpha,j} r'_{\alpha,k})$ . We can then write Eq. (16) for  $V_I$  in the compact form,

$$\begin{aligned} V_I &= -\frac{1}{2} M (\boldsymbol{\omega} \times \mathbf{R})^2 - M \dot{\mathbf{R}} \cdot (\boldsymbol{\omega} \times \mathbf{R}) \\ &- \frac{1}{2} \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\Omega}. \end{aligned} \quad (18)$$

Here,  $M = \sum m_i$  is the total mass of the rigid body. Note that even though the third term in Eq. (18) does not depend explicitly on the angular velocity of the rigid body, it depends on the orientation of its inertia tensor, except when it is a scalar (spherical top).

If we write the kinetic energy of the rigid body in the usual form,<sup>13</sup>

$$T_I = \frac{1}{2} \boldsymbol{\Omega} \cdot \mathcal{I} \cdot \boldsymbol{\Omega} + \frac{1}{2} M \dot{\mathbf{R}}^2, \quad (19)$$

we can write the Lagrangian as

$$\mathcal{L} = T_I - V_I - V_{\text{int}} \quad (20a)$$

$$= \frac{1}{2} (\boldsymbol{\Omega} + \boldsymbol{\omega}) \cdot \mathcal{I} \cdot (\boldsymbol{\Omega} + \boldsymbol{\omega}) + \frac{1}{2} M (\dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{R})^2 - V_{\text{int}}. \quad (20b)$$

Note that for a rotating reference frame with its origin at the center of mass of the rigid body, the inertial potential is a linear function of the angular velocity,

$$V_I = -\frac{1}{2} \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\Omega}. \quad (21)$$

These results provide a useful Lagrangian description of rigid bodies in rotating frames, as well as an interesting analogy between the corresponding inertial and electromagnetic forces. For instance, the first two terms in Eq. (18) can be identified with an electromagnetic field of symmetric gauge (with  $\mathbf{B} = 2\boldsymbol{\omega}$ ) acting on a single particle of charge  $Q = M$  located at the center of mass. The other two terms in Eq. (18) can be related to electric quadrupole and magnetic dipole moments. The analogy  $q_i \leftrightarrow m_i$  can be extended to identify the moment of inertia with the electric quadrupole moment and the angular momentum,  $\mathbf{L} = \mathcal{I} \cdot \boldsymbol{\Omega}$ , with the magnetic dipole moment,  $\boldsymbol{\mu} \leftrightarrow \mathbf{L}/2$ , so that  $\boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\Omega} = \boldsymbol{\mu} \cdot \mathbf{B}$ . There is no

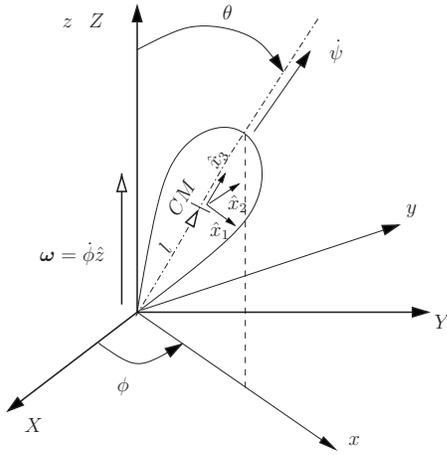


Fig. 1. Spinning top and Euler angles convention.

electric dipole moment because the expansion is about the center of mass.

## VI. EXAMPLES

### A. The gyrocompass

We first analyze the motion of a gyrocompass, that is, an electrically powered spinning wheel mounted on gimbals, which finds true north and is not affected by surrounding metals. An early version of this gyroscope was patented in 1885 by van den Bos, but it was not until almost two decades later that a working model was constructed by Anschütz-Kaempfe (1872–1931).<sup>17</sup> The successful tests run by the German Navy attracted the attention of Sperry (1860–1930), whose patenting of an improved version led to a legal battle by Anschütz-Kaempfe. Einstein, who was appointed as expert by the Royal District Court in Berlin, gave a report favorable to Anschütz-Kaempfe's claims of primacy.<sup>17</sup>

Assume that the gyrocompass rotates around its symmetry axis with moment of inertia  $I_3$  in a reference frame fixed to the surface of Earth. Because its angular velocity  $\dot{\psi}$  is much larger than the angular velocity  $|\omega|$  of Earth, we can approximate Eq. (18) as

$$V_1 \approx -I_3 \omega \dot{\psi} \cos \theta, \quad (22)$$

where  $\theta$  is the orientation of the gyrocompass about north. This inertial potential provides a restitutive force that, through friction forces, will eventually orient the compass's axis toward the north celestial pole. If we appeal to the electromagnetic analogy, the situation is similar to that of a magnetic moment within a constant magnetic field.<sup>18</sup>

### B. Kinetic energy in an inertial frame: The spinning top

The inertial potential also provides a simple way of evaluating the Lagrangian of a rotating body in an inertial reference frame. Consider the motion of a symmetrical top ( $I_1 = I_2 \neq I_3$ ) balancing on one of its extremes.<sup>13</sup> Inspection of Fig. 1 shows that in a reference frame revolving with the precession velocity  $\omega = \dot{\phi} \hat{z} = \omega \cos \theta \hat{x}_3 - \omega \sin \theta \hat{x}_1$  (not known *a priori*), the angular velocity of the body has only two contributions:  $\Omega = \dot{\theta} \hat{x}_2 + \dot{\psi} \hat{x}_3$ . Here,  $\theta$ ,  $\phi$ , and  $\psi$  are the three Eu-

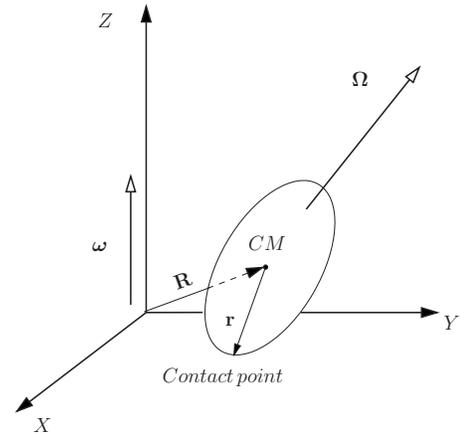


Fig. 2. Rolling body on a rotating plane with angular velocity  $\omega$ .

ler angles.<sup>13</sup> If we use Eq. (18), we obtain the inertial potential

$$V_1 = -\frac{1}{2} \omega^2 [I_3 \cos^2 \theta + (I_1 + m\ell^2) \sin^2 \theta] - I_3 \omega \dot{\psi} \cos \theta, \quad (23)$$

where  $\ell$  is the distance from the center of mass to the balancing point. Note that, although the second term in Eq. (18) vanishes, the fourth and the first and third terms contribute to  $V_1$  with linear and quadratic terms in  $\omega$ , respectively. We use  $\Omega = \dot{\theta} \hat{x}_2 + \dot{\psi} \hat{x}_3$  and write the kinetic energy in the rotating frame Eq. (19) in terms of the Euler angles as

$$T_1 = \frac{1}{2} I_3 \dot{\psi}^2 + \frac{1}{2} (I_1 + m\ell^2) \dot{\theta}^2. \quad (24)$$

This example shows how the inertial potential can be employed to decouple the motion of the rotating frame in cases where a direct analysis in an inertial frame might be cumbersome. Note in particular that it corrects the kinetic energy evaluated in the rotating frame to yield the usual expression<sup>13</sup> in an inertial frame,

$$\begin{aligned} T &= T_1 - V_1 \\ &= \frac{1}{2} I_3 (\dot{\psi} + \omega \cos \theta)^2 + \frac{1}{2} (I_1 + m\ell^2) (\dot{\theta}^2 + \omega^2 \sin^2 \theta). \end{aligned} \quad (25)$$

### C. General case: A first integral for rolling bodies

As another example, we derive a first integral for the motion of an arbitrary (strictly) convex rigid body projected onto a horizontal plane that rotates with angular velocity  $\omega = \omega \hat{z}$ . We describe its dynamics by means of the position  $\mathbf{R}$  of its center of mass and the three Euler angles ( $\phi, \theta, \psi$ ) in a reference frame rotating with the plane (see Fig. 2). Except for bodies with particularly strong symmetries such as disks or spheres,<sup>19–21</sup> the vector  $\mathbf{r}$  from the center of mass to the contact point (see Fig. 2) is not usually fixed in magnitude or direction, but is a function of the Euler angles,  $\mathbf{r}(\phi, \theta, \psi)$ . Assume that the body does not slide, but is free to roll on the surface and to rotate about the vertical axis  $\hat{z}$ . Under these circumstances its motion is constrained by<sup>8</sup>

$$\mathbf{0} = \dot{\mathbf{R}} + \boldsymbol{\Omega} \times \mathbf{r}, \quad (26)$$

where  $\boldsymbol{\Omega}$  is its angular velocity. The projection of this holonomic constraint onto the  $\hat{z}$  direction is trivial because the contact point does not move in that direction (in fact, this condition reduces the problem to five degrees of freedom). The projection of Eq. (26) onto  $\hat{x}$  and  $\hat{y}$  yields two nontrivial, nonintegrable differential constraints. These constraints are time independent and linear in the velocities  $\dot{\mathbf{R}}$  and  $\boldsymbol{\Omega}$  (and, for that reason, linear in the generalized velocities  $\dot{q}_i$ ). Therefore, if they are included in the Lagrangian by means of Lagrange multipliers,<sup>22</sup> they leave the Lagrangian time independent and quadratic in the generalized velocities,

$$\mathcal{L} = \sum_{ij} A_{ij}(q) \dot{q}_i \dot{q}_j + \sum_i B_i(q) \dot{q}_i - D(q). \quad (27)$$

Thus, according to Noether's theorem,<sup>23</sup> we have the following first integral:<sup>24</sup>

$$c = \sum_k \dot{q}_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \mathcal{L} = \sum_{ij} A_{ij}(q) \dot{q}_i \dot{q}_j + D(q). \quad (28)$$

Equation (28) shows that to construct a constant of motion we have only to remove the linear terms in  $\dot{q}$  from the Lagrangian and change the sign of the velocity-independent term. In particular, the constraints are not present in this conservation law because the corresponding terms in the Lagrangian are linear in the generalized-velocities. We write the potential Eq. (18) as

$$V_I = - \underbrace{\frac{1}{2} M(\boldsymbol{\omega} \times \mathbf{R})^2 - \frac{1}{2} \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\omega}}_{\text{velocity independent}} - \underbrace{\boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\Omega} - M \dot{\mathbf{R}} \cdot (\boldsymbol{\omega} \times \mathbf{R})}_{\text{linear in velocities}}, \quad (29)$$

and we finally obtain

$$c = \frac{1}{2} \boldsymbol{\Omega} \cdot \mathcal{I} \cdot \boldsymbol{\Omega} + \frac{1}{2} M \dot{\mathbf{R}}^2 - \frac{1}{2} M(\boldsymbol{\omega} \times \mathbf{R})^2 - \frac{1}{2} \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\omega}. \quad (30)$$

Equation (30) is the noninertial version of energy conservation. The first two terms are the kinetic energy  $T_I$  and the third is an effective centrifugal potential. The last term is an effective potential which depends on the orientation of the body, so it is nontrivial except for the special case of spherical symmetry.

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